

Conformal Modules

by

Shun-Jen Cheng[†] and Victor G. Kac^{††}

[†]Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan
chengsj@mail.ncku.edu.tw

^{††}Department of Mathematics, MIT, Cambridge, MA 02139, USA
kac@math.mit.edu

Abstract. In this paper we study a class of modules over infinite-dimensional Lie (super)algebras, which we call conformal modules. In particular we classify and construct explicitly all irreducible conformal modules over the Virasoro and the $N=1$ Neveu-Schwarz algebras, and over the current algebras.

0. Introduction

Conformal module is a basic tool for the construction of free field realization of infinite-dimensional Lie (super)algebras in conformal field theory. This is one of the reasons to classify and construct such modules. In the present paper we solve this problem under the irreducibility assumption for the Virasoro and the Neveu-Schwarz algebras, for the current algebras and their semidirect sums. Since complete reducibility does not hold for conformal modules, one has to discuss the extension problem. This problem is solved in [1].

The basic idea of our approach is to use three (more or less) equivalent languages. The first is the language of local formal distributions, the second is the language of modules over conformal algebras, and the third is the language of conformal modules over the annihilation subalgebras. The problem is solved using the third language by means of the crucial Lemma 3.1. Note that conformal modules over Lie algebras of Cartan type were studied in [6], where, in particular, a proof of Corollary 3.3 is contained.

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This paper is organized as follows. In Section 1 the concepts of a Lie (super)algebra of formal distributions and of a conformal (super)algebra are recalled. They are more or less equivalent notions. In Section 2 we introduce conformal modules over a Lie (super)algebra of formal distributions and clarify their connections with modules over the corresponding conformal (super)algebra. We then show that modules over a conformal (super)algebra are in 1-1 correspondence with modules over the annihilation subalgebra of the associated Lie (super)algebra of formal distributions. At the end of Section 2 examples of conformal modules over the Virasoro, the current and Neveu-Schwarz algebras and their various semidirect sums are constructed. In Section 3 we first prove the key lemma (Lemma 3.1) and with its help classify all irreducible finite conformal modules over the annihilation subalgebra of the above-mentioned Lie (super)algebras. The main result, stated in Theorem 3.2, which describes all finite irreducible modules over the conformal (super)algebras in question (hence all irreducible finite conformal modules over the corresponding Lie (super)algebras), is then immediate.

Roughly speaking, the main claim of the present paper is that all non-trivial modules over current, Virasoro, $N=1$ superconformal algebras and their semidirect sums are non-degenerate. For $N>1$ superconformal algebras interesting degeneracies occur in some non-trivial conformal modules. These effects are studied in [5].

1. Preliminaries on local formal distributions and conformal superalgebras

A *formal distribution* (usually called a field by physicists) with coefficients in a complex vector space U is a generating series of the form:

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} \in U$ and z is an indeterminate.

Two formal distributions $a(z)$ and $b(z)$ with coefficients in a Lie superalgebra \mathfrak{g} are called (mutually) *local* if for some $N \in \mathbb{Z}_+$ one has:

$$(z - w)^N [a(z), b(w)] = 0. \tag{1.1}$$

Introducing the *formal delta function*

$$\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n,$$

we may write a condition equivalent to (1.1):

$$[a(z), b(w)] = \sum_{j=0}^N (a_{(j)}b)(w) \partial_w^{(j)} \delta(z - w), \quad (1.2)$$

(here $\partial_w^{(j)}$ stands for $\frac{1}{j!} \frac{\partial^j}{\partial w^j}$) for some formal distributions $(a_{(j)}b)(w)$ ([3], Theorem 2.3), which are uniquely determined by the formula

$$(a_{(j)}b)(w) = \text{Res}_z (z - w)^j [a(z), b(w)]. \quad (1.3)$$

Formula (1.3) defines a \mathbb{C} -bilinear product $a_{(j)}b$ for each $j \in \mathbb{Z}_+$ on the space of all formal distributions with coefficients in \mathfrak{g} .

Note also that the space (over \mathbb{C}) of all formal distributions with coefficients in \mathfrak{g} is a (left) module over $\mathbb{C}[\partial_z]$, where the action of ∂_z is defined in the obvious way, so that $\partial_z a(z) = \sum_n (\partial a)_{(n)} z^{-n-1}$, where $(\partial a)_{(n)} = -n a_{(n-1)}$.

The Lie superalgebra \mathfrak{g} is called a *Lie superalgebra of formal distributions* if there exists a family \mathfrak{F} of pairwise local formal distributions whose coefficients span \mathfrak{g} . In such a case we say that the family \mathfrak{F} *spans* \mathfrak{g} . We will write $(\mathfrak{g}, \mathfrak{F})$ to emphasize the dependence on \mathfrak{F} .

The simplest example of a Lie superalgebra of formal distributions is the *current superalgebra* $\tilde{\mathfrak{g}}$ associated to a finite-dimensional Lie superalgebra \mathfrak{g} :

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

It is spanned by the following family of pairwise local formal distributions $a \in \mathfrak{g}$:

$$a(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) z^{-n-1}.$$

Indeed, it is immediate to check that

$$[a(z), b(w)] = [a, b](w) \delta(z - w).$$

The simplest example beyond current algebras is the (centerless) *Virasoro algebra*, the Lie algebra \mathfrak{V} with basis L_n ($n \in \mathbb{Z}$) and commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}.$$

It is spanned by the local formal distribution $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, since one has:

$$[L(z), L(w)] = \partial_w L(w) \delta(z - w) + 2L(w) \partial_w \delta(z - w). \quad (1.4)$$

The next important example is the semidirect sum $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$ with the usual commutation relations between \mathfrak{V} and $\tilde{\mathfrak{g}}$:

$$[L_m, a \otimes t^n] = -na \otimes t^{m+n},$$

which is equivalent to

$$[L(z), a(w)] = \partial_w a(w) \delta(z - w) + a(w) \partial_w \delta(z - w).$$

Given a Lie superalgebra of formal distributions $(\mathfrak{g}, \mathfrak{F})$, we may always include \mathfrak{F} in the minimal family \mathfrak{F}^c of pairwise local distributions which is closed under ∂ and all products (1.3) ([3], Section 2.7). Then \mathfrak{F}^c is a *conformal superalgebra* with respect to the products (1.3). Its definition is given below [3]:

A *conformal superalgebra* is a left $(\mathbb{Z}_2\text{-graded}) \mathbb{C}[\partial]$ -module R with a \mathbb{C} -bilinear product $a_{(n)}b$ for each $n \in \mathbb{Z}_+$ such that the following axioms hold ($a, b, c \in R; m, n \in \mathbb{Z}_+$ and $\partial^{(j)} = \frac{1}{j!} \partial^j$):

$$(C0) \quad a_{(n)}b = 0, \text{ for } n \gg 0,$$

$$(C1) \quad (\partial a)_{(n)}b = -na_{(n-1)}b,$$

$$(C2) \quad a_{(n)}b = (-1)^{\deg a \deg b} \sum_{j=0}^{\infty} (-1)^{j+n+1} \partial^{(j)}(b_{(n+j)}a),$$

$$(C3) \quad a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c + (-1)^{\deg a \deg b} b_{(n)}(a_{(m)}c).$$

Of course, conformal algebra coincides with its even part, i.e. $\deg a = 0$ for all $a \in R$ in this case. Note also the following consequence of (C1) and (C2):

$$(C2') \quad a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b,$$

hence ∂ is a derivation of all products (1.3).

Conversely, assuming for simplicity (cf. Lemma 2.2b) that $R = \oplus_{i \in I} \mathbb{C}[\partial]a^i$ is a free (as a $\mathbb{C}[\partial]$ -module) conformal superalgebra, we may associate to R a Lie superalgebra of formal distributions $(\mathfrak{g}(R), \mathfrak{F})$ with basis $a_{(m)}^i$ ($i \in I, m \in \mathbb{Z}$) and $\mathfrak{F} = \{a^i(z) = \sum_n a_{(n)}^i z^{-n-1}\}_{i \in I}$ with bracket (cf. (1.2)):

$$[a^i(z), a^j(w)] = \sum_{k \in \mathbb{Z}_+} (a_{(k)}^i a^j)(w) \partial_w^{(k)} \delta(z-w), \quad (1.5)$$

so that $\mathfrak{F}^c = R$.

Formula (1.5) is equivalent to the following commutation relations ($m, n \in \mathbb{Z}; i, j \in I$):

$$[a_{(m)}^i, a_{(n)}^j] = \sum_{k \in \mathbb{Z}_+} \binom{m}{k} (a_{(k)}^i a^j)_{(m+n-k)}. \quad (1.6)$$

It follows that the \mathbb{C} -span of all $a_{(n)}^i$ with $n \in \mathbb{Z}_+$ is a subalgebra of the Lie superalgebra $\mathfrak{g}(R)$. We denote this subalgebra by $\mathfrak{g}(R)_+$ and call it the *annihilation subalgebra*.

For example $\mathfrak{V}_+ = \sum_{j \geq -1} \mathbb{C}L_j$ and $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t]$.

The simplest examples of a conformal superalgebra is the *current conformal superalgebra* associated to a finite-dimensional Lie superalgebra \mathfrak{g} :

$$R(\tilde{\mathfrak{g}}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathfrak{g},$$

with the products defined by:

$$a_{(0)}b = [a, b], \quad a_{(j)}b = 0, \quad \text{for } j > 0, \quad a, b \in \mathfrak{g},$$

and the *Virasoro conformal algebra* $R(\mathfrak{V}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} L$ with products (cf. (1.4)):

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(j)}L = 0, \quad \text{for } j > 1.$$

Their semidirect sum is $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ with additional non-zero products $L_{(0)}a = \partial a$ and $L_{(1)}a = a$, for $a \in \mathfrak{g}$. These examples are the conformal superalgebras associated to the Lie superalgebras of formal distributions described above.

The simplest superextension of the Virasoro algebra is the well-known (centerless) Neveu-Schwarz algebra \mathfrak{N} which, apart from even basis elements L_n , has odd basis elements G_r , $r \in \frac{1}{2} + \mathbb{Z}$, with commutation relations:

$$[G_r, L_n] = (r - \frac{n}{2})G_{r+n}, \quad [G_r, G_s] = 2L_{r+s}.$$

The corresponding annihilation subalgebra in this case is $\mathfrak{N}_+ = \sum_{n \geq -1} \mathbb{C} L_n + \sum_{r \geq -\frac{1}{2}} \mathbb{C} G_r$. The conformal superalgebra, associated to \mathfrak{N} , is $R(\mathfrak{N}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} L + \mathbb{C}[\partial] \otimes_{\mathbb{C}} G$ with additional non-zero products:

$$L_{(0)}G = \partial G, \quad G_{(0)}L = \frac{1}{2}\partial G, \quad L_{(1)}G = G_{(1)}L = \frac{3}{2}G, \quad G_{(0)}G = 2L.$$

Other examples treated in this paper are *supercurrent algebras*

$$\tilde{\mathfrak{g}}_{\text{super}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}, \theta],$$

where θ is an odd indeterminate. The generating family \mathfrak{F} of pairwise local formal distributions consists of currents $a(z)$ ($a \in \mathfrak{g}$) introduced above and supercurrents

$$a^\theta(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n \theta) z^{-n-1}.$$

Of course its associated conformal superalgebra is $R(\tilde{\mathfrak{g}}_{\text{super}}) = \mathbb{C}[\partial] \otimes (\mathfrak{g} \oplus \mathfrak{g}^\theta)$, where \mathfrak{g}^θ is an identical copy of \mathfrak{g} , but with reversed parity. $R(\tilde{\mathfrak{g}}_{\text{super}})$ extends $R(\tilde{\mathfrak{g}})$ by the additional non-trivial product

$$a_{(0)}b^\theta = [a, b]^\theta, \quad a, b \in \mathfrak{g}.$$

The final example treated in this paper is the semidirect sum $\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}$, which is defined by letting $L_n = -t^n(t \frac{\partial}{\partial t} + \frac{n+1}{2}\theta \frac{\partial}{\partial \theta})$ and $G_r = -t^{r+\frac{1}{2}}(\theta \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta})$ for $n \in \mathbb{Z}$ and $r \in \frac{1}{2} + \mathbb{Z}$. Its corresponding conformal superalgebra $R(\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}) = R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\text{super}})$ has the following additional non-trivial products:

$$L_{(0)}a^\theta = \partial a^\theta, \quad L_{(1)}a^\theta = \frac{1}{2}a^\theta, \quad G_{(0)}a^\theta = a, \quad G_{(0)}a = \partial a^\theta, \quad G_{(1)}a = a^\theta.$$

2. Preliminaries on conformal modules

Let $(\mathfrak{g}, \mathfrak{F})$ be a Lie superalgebra of formal distributions, and let V be a \mathfrak{g} -module. We say that a formal distribution $a(z) \in \mathfrak{F}$ and a formal distribution $v(z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$ with coefficients in V are *local* if

$$(z-w)^N a(z)v(w) = 0, \quad \text{for some } N \in \mathbb{Z}_+. \quad (2.1)$$

It follows from [3] Section 2.3 that (2.1) is equivalent to

$$a(z)v(w) = \sum_{j=0}^{N-1} (a_{(j)}v)(w) \partial_w^{(j)} \delta(z-w), \quad (2.2)$$

for some formal distributions $(a_{(j)}v)(w)$ with coefficients in V , which are uniquely determined by the formula

$$(a_{(j)}v)(w) = \text{Res}_z (z-w)^j a(z)v(w).$$

Formula (2.2) is obviously equivalent to

$$a_{(m)}v_{(n)} = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}v)_{(m+n-j)}. \quad (2.3)$$

Example 2.1. Consider the following representation of the (centerless) Virasoro algebra in the vector space V with basis $v_{(n)}$, $n \in \mathbb{Z}$, over \mathbb{C}

$$L_m v_{(n)} = ((\Delta - 1)(m + 1) - n)v_{(m+n)} + \alpha v_{(m+n+1)},$$

where $\Delta, \alpha \in \mathbb{C}$. In terms of formal distributions $L(z)$ and $v(z)$ this can be written as follows:

$$L(z)v(w) = (\partial + \alpha)v(w)\delta(z-w) + \Delta v(w)\partial_w \delta(z-w). \quad (2.4)$$

Hence $L(z)$ and $v(z)$ are local.

Suppose that V is spanned over \mathbb{C} by the coefficients of a family E of formal distributions such that all $a(z) \in \mathfrak{F}$ are local with respect to all $v(z) \in E$. Then we call (V, E) a *conformal module over $(\mathfrak{g}, \mathfrak{F})$* .

The following is a representation-theoretic analogue (and a generalization) of Dong's lemma (see [3], Section 3.2).

Lemma 2.1. [4] *Let V be a module over a Lie superalgebra \mathfrak{g} , let $a(z)$ and $b(z)$ (respectively $v(z)$) be formal distributions with coefficients in \mathfrak{g} (respectively in V). Suppose that all pairs (a, b) , (a, v) and (b, v) are local. Then the pairs $(a_{(j)}b, v)$ and $(a, b_{(j)}v)$ are local for all $j \in \mathbb{Z}_+$.*

This lemma shows that the family E of a conformal module (V, E) over $(\mathfrak{g}, \mathfrak{F})$ can always be included in a larger family E^c which is still local with respect to \mathfrak{F} , hence to \mathfrak{F}^c , and such that $\partial E^c \subset E^c$ and $a_{(j)}E^c \subset E^c$ for all $a \in \mathfrak{F}$ and $j \in \mathbb{Z}_+$.

It is straightforward to check the following properties for $a, b \in \mathfrak{F}$ and $v \in E^c$:

$$[a_{(m)}, b_{(n)}]v = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}v, \quad (2.5)$$

$$(\partial a)_{(n)}v = [\partial, a_{(n)}]v = -na_{(n-1)}v. \quad (2.6)$$

(Here $[\cdot, \cdot]$ is the bracket of operators on E^c .) It follows from (2.5) (by induction on m) and (2.6) that $a_{(j)}E^c \subset E^c$ for all $a \in \mathfrak{F}^c$ and $j \in \mathbb{Z}_+$.

Thus any conformal module (V, E) over a Lie superalgebra of formal distributions $(\mathfrak{g}, \mathfrak{F})$ gives rise to a module $M = E^c$ over the conformal superalgebra $R = \mathfrak{F}^c$, defined as follows. It is a (left) \mathbb{Z}_2 -graded $\mathbb{C}[\partial]$ -module equipped with a family of \mathbb{C} -linear maps $a \rightarrow a_{(n)}^M$ of R to $\text{End}_{\mathbb{C}} M$, for each $n \in \mathbb{Z}_+$, such that the following properties hold (cf. (2.5) and (2.6)) for $a, b \in R$ and $m, n \in \mathbb{Z}_+$:

- (M0) $a_{(n)}^M v = 0$, for $v \in M$ and $n \gg 0$,
- (M1) $[a_{(m)}^M, b_{(n)}^M] = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}^M$,
- (M2) $(\partial a)_{(n)}^M = [\partial, a_{(n)}^M] = -na_{(n-1)}^M$.

Conversely, suppose that a conformal superalgebra $R = \oplus_{i \in I} \mathbb{C}[\partial]a^i$ is a free $\mathbb{C}[\partial]$ -module and consider the associated Lie superalgebra of formal distributions $(\mathfrak{g}(R), \mathfrak{F})$ (see Section 1). Let M be a module over the conformal superalgebra R and suppose (cf. Lemma 2.2b) that M is a free $\mathbb{C}[\partial]$ -module with $\mathbb{C}[\partial]$ -basis $\{v^\alpha\}_{\alpha \in J}$. This gives rise to a conformal $\mathfrak{g}(R)$ -module $V(M)$ with basis $v_{(n)}^\alpha$, where $\alpha \in J$ and $n \in \mathbb{Z}$, defined by (cf. (2.2)):

$$a^i(z)v^\alpha(w) = \sum_{j \in \mathbb{Z}_+} (a_{(j)}^i v^\alpha)(w) \partial_w^{(j)} \delta(z-w). \quad (2.7)$$

A conformal module (V, E) (respectively module M) over a Lie superalgebra of formal distributions $(\mathfrak{g}, \mathfrak{F})$ (respectively over a conformal superalgebra R) is called *finite*, if E^c (respectively M) is a finitely generated $\mathbb{C}[\partial]$ -module.

A conformal module (V, E) over $(\mathfrak{g}, \mathfrak{F})$ is called *irreducible* if there is no non-trivial invariant subspace which contains all $v_{(n)}$, $n \in \mathbb{Z}$, for some non-zero $v \in E^c$. Clearly a conformal module is irreducible if and only if the associated module E^c over the conformal superalgebra \mathfrak{F}^c is irreducible (in the obvious sense).

The above discussions, combined with the following lemma, reduce the classification of finite conformal modules over a Lie superalgebra of formal distributions $(\mathfrak{g}, \mathfrak{F})$ to the classification of finite modules over the corresponding conformal superalgebra.

Lemma 2.2. [4] (a) *Let M be a module over a conformal superalgebra R and let $v \in M$ be such that $\partial v = \lambda v$ for some $\lambda \in \mathbb{C}$. Then v is an invariant, i.e. $R_{(m)}v = 0$ for all $m \in \mathbb{Z}_+$.*

(b) *Suppose that M is a finite module over a conformal superalgebra and suppose that M has no non-zero invariants. Then M is a free $\mathbb{C}[\partial]$ -module.*

Remark 2.1. Given a module M over a conformal superalgebra R , we may change its structure as a $\mathbb{C}[\partial]$ -module by replacing ∂ by $\partial + A$, where A is an endomorphism over \mathbb{C} of M which commutes with all $a_{(n)}^M$ (this will not affect axiom (M2)).

Note that the maps $a \rightarrow a_{(n)}$ of R to $\text{End}_{\mathbb{C}} R$ defines an R -module, called the *adjoint module*.

Example 2.1 gives a 2-parameter family of (irreducible) modules over the Virasoro conformal algebra. Note also that the well-known family of graded Virasoro modules given by $L_m v_{(n)} = ((\Delta - 1)(m + 1) - n + \alpha) v_{(m+n)}$ is conformal, but is finite if and only if $\alpha = 0$.

The following simple observation, which follows from definitions, is fundamental for representation theory of conformal superalgebras.

Proposition 2.1. *Consider the Lie superalgebra of formal distributions $(\mathfrak{g}(R), \mathfrak{F})$ defined by (1.5) and let $\mathfrak{g}(R)_+$ be the annihilation subalgebra of $\mathfrak{g}(R)$. Denote by $\mathfrak{g}(R)^+$ the semidirect product of the 1-dimensional Lie subalgebra $\mathbb{C}\partial$ and the ideal $\mathfrak{g}(R)_+$ with the action of ∂ on $\mathfrak{g}(R)_+$ given by $\partial(a_{(n)}^i) = -na_{(n-1)}^i$. Then a module M over the conformal superalgebra R is precisely a $\mathfrak{g}(R)^+$ -module M (over \mathbb{C}) such that*

$$a_{(n)}^i v = 0, \quad \text{for each } v \in M \text{ and } n \gg 0. \quad (2.8)$$

Corollary 2.1. *Let $R = \oplus_{i \in I} \mathbb{C}[\partial] a^i$ be a conformal superalgebra and $M = \oplus_{j \in J} \mathbb{C}[\partial] v^j$ be a free $\mathbb{C}[\partial]$ -module. Then, given $a_{(n)}^i v^j \in M$ for all $i \in I, j \in J, n \in \mathbb{Z}_+$, which is 0 for $n \gg 0$, we may extend uniquely the action of $a_{(n)}^i$ to the all of R on M using (M2). Suppose that (M1) holds for all $a = a_{(m)}^i, b = a_{(n)}^j$. Then M is an R -module.*

Using Proposition 2.1 and Corollary 2.1, one can construct large families of finite modules over conformal superalgebras, hence corresponding modules over Lie superalgebras of formal distributions.

In conclusion we will list more examples of modules over conformal superalgebras. In Section 3 the irreducible ones listed below will turn out to exhaust the list of all irreducible finite conformal modules over these conformal superalgebras.

Example 2.2. Let $\mathfrak{V}_0 = \sum_{j \geq 0} \mathbb{C}L_j$ and consider a representation π of the Lie algebra \mathfrak{V}_0 in a finite-dimensional (over \mathbb{C}) vector space U . Let A be an endomorphism of U commuting with all $\pi(L_j)$ ($j \in \mathbb{Z}_+$). Then $\mathbb{C}[\partial] \otimes U$ is a finite module over the conformal algebra \mathfrak{V} defined by the following formulas ($u \in U$):

$$L_{(0)}u = (\partial + A)u, \quad L_{(j)}u = \pi(L_{j-1})u, \text{ for } j \geq 1.$$

For example we can take $\pi(L_0) = B$, where B is an endomorphism of U commuting with A . Then

$$L_{(0)}u = \partial u + Au, \quad L_{(1)}u = Bu, \quad L_{(j)}u = 0, \quad j \geq 1,$$

defines a finite module over the centerless conformal algebra \mathfrak{V} , which we denote by $M_{\mathfrak{V}}(A, B)$. Taking $\dim U = 1$, $A = \alpha$ and $B = \Delta$, where $\alpha, \Delta \in \mathbb{C}$, gives Example 2.1.

Translating back to the language of Lie algebras of formal distributions, Example 2.2 gives the following family of finite conformal modules over the Virasoro algebra in the space $U \otimes \mathbb{C}[t, t^{-1}]$ (we let, as usual, $u_{(n)} = u \otimes t^n$):

$$L_m u_{(n)} = (Au)_{(m+n+1)} - (m+n+1)u_{(m+n)} + \sum_{j=0}^{\infty} \binom{m+1}{j+1} (\pi(L_j)u)_{(m+n-j)}.$$

The above special case defined by a pair (A, B) of the commuting operators in U is given by:

$$L_m u_{(n)} = (Au)_{(m+n+1)} + ((m+1)Bu - (m+n+1)u)_{(m+n)}.$$

We keep the notation $M_{\mathfrak{V}}(A, B)$ for this \mathfrak{V} -module. Note that the module $M_{\mathfrak{V}}(A, B)$ is irreducible if and only if the \mathfrak{V}_0 -module is irreducible and non-trivial, i.e. if and only if

$\dim U = 1$ and $B = \Delta \neq 0$. We denote these \mathfrak{V} -modules again by $M_{\mathfrak{V}}(\alpha, \Delta)$, $\alpha, \Delta \in \mathbb{C}$ and $\Delta \neq 0$.

Remark 2.2. (cf. Remark 2.1 and Example 2.2.) Suppose that the annihilation subalgebra \mathfrak{g}_+ of the Lie algebra $\mathfrak{g}^+ = \mathbb{C}[\partial] \ltimes \mathfrak{g}_+$, contains an element L_{-1} such that $\text{ad} L_{-1} = \partial$ on \mathfrak{g}_+ . Then \mathfrak{g}^+ is a direct sum of ideals $\mathbb{C}(\partial - L_{-1})$ and \mathfrak{g}_+ . Hence in this case the study of conformal modules reduces to the study of modules over \mathfrak{g}_+ satisfying (2.8). This remark applies to all cases except for the current and the supercurrent algebras.

Example 2.3. Consider the current Lie superalgebra $\tilde{\mathfrak{g}}$ and the associated conformal superalgebra $R(\tilde{\mathfrak{g}})$. Let π be a representation of $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$ in a finite-dimensional vector space U , such that $(t^n \otimes \mathfrak{g})U = 0$ for $n \gg 0$. This defines on the space $U \otimes \mathbb{C}[t, t^{-1}]$ the structure of a conformal module over $\tilde{\mathfrak{g}}$ by the formula:

$$(a \otimes t^m)u_{(n)} = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (\pi(a \otimes t^j)u)_{(m+n-j)}.$$

A special case of this construction is to take a finite-dimensional representation π of the Lie superalgebra \mathfrak{g} in a finite-dimensional vector space U and extend it to $\tilde{\mathfrak{g}}_+$ by letting $\mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially. Then we have $(a \otimes t^m)u_{(n)} = (\pi(a)u)_{(m+n)}$. Translating back to the language of modules over the conformal algebra $R(\tilde{\mathfrak{g}})$ we obtain the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ with

$$a_{(0)}u = \pi(a)u, \quad a_{(n)}u = 0 \quad \text{for } n > 0, \quad \text{where } u \in U$$

We will denote this module by $M_{\tilde{\mathfrak{g}}}(\pi)$. It is irreducible if and only if π is irreducible. In this case we will denote the module by $M_{\tilde{\mathfrak{g}}}(\Lambda)$, where Λ is the highest weight of U .

Example 2.4. Consider the $N=1$ Neveu-Schwarz algebra \mathfrak{N} with associated conformal superalgebra $R(\mathfrak{N})$. Let \mathfrak{N}_+ denote the corresponding annihilation subalgebra. Consider a finite-dimensional representation (π, U) of \mathfrak{N}_0 , the subalgebra of \mathfrak{N}_+ spanned by elements of non-negative modes. Let U^θ denote an identical copy of U with reversed parity. Then the following gives a structure of a module over $R(\mathfrak{N})$ on the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes (U \oplus U^\theta)$:

$$\begin{aligned} L_{(0)}u &= (\partial + A)u, & L_{(j)}u &= \pi(L_{j-1})u, & L_{(0)}u^\theta &= (\partial + A)u^\theta, \\ L_{(1)}u^\theta &= (\pi(L_0)u)^\theta + \frac{1}{2}u^\theta, & L_{(j+1)}u^\theta &= (\pi(L_j)u)^\theta + \frac{j+1}{2}\pi(G_{j-\frac{1}{2}})u, & G_{(0)}u &= u^\theta, \end{aligned}$$

$$G_{(j)}u = \pi(G_{j-\frac{1}{2}})u, \quad G_{(0)}u^\theta = (\partial u + Au), \quad G_{(j)}u^\theta = (\pi(G_{j-\frac{1}{2}})u)^\theta + 2\pi(L_{j-1})u,$$

where $j \geq 1$, $u \in U$, and A is an operator acting on U , commuting with all $\pi(\mathfrak{N}_0)$. In particular let $U = \mathbb{C}u$ be the one-dimensional \mathfrak{N}_0 -module with action $L_0u = \Delta u$, with $0 \neq \Delta \in \mathbb{C}$, all other generators acting trivially. Then for $A = \alpha \in \mathbb{C}$ arbitrary, we obtain an irreducible module over $R(\mathfrak{N})$. Hence as in the case of the Virasoro algebra we get a 2-parameter family of finite irreducible conformal modules. Denote this family by $M_{\mathfrak{N}}(\alpha, \Delta)$.

Example 2.5. Consider the supercurrent algebra $\tilde{\mathfrak{g}}_{\text{super}}$. Let $R(\tilde{\mathfrak{g}}_{\text{super}})$, $\tilde{\mathfrak{g}}_{\text{super}+}$ be as usual. Let (π, U) be a finite-dimensional representation of $\tilde{\mathfrak{g}}_{\text{super}+}$. We obtain a module over $R(\tilde{\mathfrak{g}}_{\text{super}})$ as in the case of current algebra by setting for $n \geq 0$:

$$a_{(n)}u = \pi(a \otimes t^n)u, \quad a_{(n)}^\theta u = \pi(a \otimes t^n \theta)u, \quad a \in \mathfrak{g}, \quad u \in U.$$

Denote these modules by $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\pi)$. In the special case when U is a finite-dimensional irreducible representation of \mathfrak{g} of highest weight $\Lambda \neq 0$, extended to $\tilde{\mathfrak{g}}_{\text{super}+}$ trivially, the associated module over $R(\tilde{\mathfrak{g}}_{\text{super}})$ is irreducible and finite. We denote this module by $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\Lambda)$.

3. The key lemma and classification of finite irreducible conformal modules

Let $(\mathfrak{g}, \mathfrak{F})$ be a Lie superalgebra of formal distributions. For each $N \in \mathbb{Z}_+$ let

$$\mathfrak{g}_N = \sum_{a \in \mathfrak{F}, n \geq N} \mathbb{C}a_{(n)}.$$

Suppose that $(\mathfrak{g}, \mathfrak{F})$ is *regular* [3], i.e. there exists a derivation ∂ of \mathfrak{g} such that $\partial(a_{(n)}) = -na_{(n-1)}$ for all $a \in \mathfrak{F}$ and $n \in \mathbb{Z}$. Obviously, $(\mathfrak{g}(R), \mathfrak{F})$, where R is a conformal superalgebra, is regular, hence all examples considered above are regular. Then $\mathfrak{g}_+ = \mathfrak{g}_0$ is the annihilation subalgebra, which is ∂ -invariant and, due to Proposition 2.1, we have to study representations of the Lie superalgebra $\mathfrak{g}^+ = \mathbb{C}\partial \ltimes \mathfrak{g}_+$, called the *extended annihilation subalgebra*. This leads us to consider the following more abstract situation.

Let \mathfrak{L} be a Lie superalgebra over \mathbb{C} with a distinguished element ∂ and a descending sequence of subspaces $\mathfrak{L} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_n \supset \cdots$, such that $[\partial, \mathfrak{L}_k] = \mathfrak{L}_{k-1}$, for

all $k > 0$. Let V be an \mathfrak{L} -module such that given any $v \in V$ there exists a non-negative integer N (depending on v) such that $\mathfrak{L}_N v = 0$. We will call such \mathfrak{L} -modules *conformal \mathfrak{L} -modules*. A conformal \mathfrak{L} -module is called *finite* if it is finitely generated as a $\mathbb{C}[\partial]$ -module. Our goal is to describe irreducible finite conformal \mathfrak{L} -modules.

Let V be a conformal \mathfrak{L} -module. Let $V_n = \{v \in V \mid \mathfrak{L}_n v = 0\}$, and let N be the minimal non-negative integer such that $V_N \neq 0$ (it exists by definition). Let $U = V_N$. Now we can state the key lemma.

Lemma 3.1. *Suppose that $N \geq 1$. Then $\mathbb{C}[\partial]U = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ and hence $\mathbb{C}[\partial]U \cap U = U$. In particular U is a finite-dimensional vector space if V is finite.*

Proof. Let L_a and R_a denote the left and right multiplication by the element a , respectively. Using $R_a = L_a - \text{ada}$ and the binomial formula, we get the following well-known formula in any associative algebra A ,

$$ga^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} (-\text{ada})^j g, \quad a, g \in A.$$

Let $\{v_i\}$, $i \in I$, be a \mathbb{C} -linearly independent set of vectors in U generating the $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial]U$. Suppose that $v = \sum_{i=1}^n p_i(\partial)v_i = 0$, where $p_i(\partial) \in \mathbb{C}[\partial]$, and not all $p_i(\partial) = 0$. Let m be the maximal degree of the $p_i(\partial)$'s. We write $p_i(\partial) = \sum_{j=0}^m c_{ij}\partial^j$, $c_{ij} \in \mathbb{C}$, so that we have $c_{im} \neq 0$, for some i . We have, since $N \geq 1$:

$$\begin{aligned} \mathfrak{L}_{N+m-1}\partial^k &= \sum_{j=0}^k \binom{k}{j} \partial^{k-j} (\text{ad}\partial)^j (\mathfrak{L}_{N+m-1}) \\ &= \sum_{j=0}^k \binom{k}{j} \partial^{k-j} \mathfrak{L}_{N+m-1-j}. \end{aligned}$$

We have therefore

$$0 = \mathfrak{L}_{N+m-1}v = \sum_{i=1}^n c_{im} \mathfrak{L}_{N-1}v_i = \mathfrak{L}_{N-1} \left(\sum_{i=1}^n c_{im} v_i \right).$$

Since $\sum_{i=1}^n c_{im} v_i \neq 0$, this contradicts the minimality of N . Hence all the $p_i(\partial) = 0$, proving the lemma. \square

Theorem 3.1. Assume that $[\mathfrak{L}_0, \mathfrak{L}_i] \subset \mathfrak{L}_i$ for all $i \geq 0$ and that $\mathfrak{L} = \mathbb{C}\partial + \mathbb{C}[\partial, \partial] + \mathfrak{L}_0$. Let V be a non-trivial irreducible conformal \mathfrak{L} -module. Then either $V \cong \text{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$, where U is an irreducible \mathfrak{L}_0 -module (and $\dim U < \infty$ if V is finite), or else, provided that \mathfrak{L}_0 is an ideal of \mathfrak{L} , V can be a (non-trivial) finite-dimensional irreducible $\mathfrak{L}/\mathfrak{L}_0$ -module.

Proof. We continue to use the notation above. Clearly U is \mathfrak{L}_0 -invariant. First assume that $N \geq 1$. By Lemma 3.1 we see that V contains an \mathfrak{L} -submodule isomorphic to $\text{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$. Hence $V \cong \text{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$. Clearly U must be irreducible over \mathfrak{L}_0 in order for V to be irreducible. Conversely the same argument as in the proof of Lemma 3.1 shows that by applying \mathfrak{L}_k , for a suitable k , to a non-zero element of the form $\sum_{i=1}^n q_i(\partial)v_i$, where $q_i(\partial) \in \mathbb{C}[\partial]$ and $v_i \in U$, one obtains a non-zero element in U . This implies that such induced modules are irreducible.

Now suppose that $N = 0$. Let u be a non-zero vector in U . Then $U(\mathfrak{L})u = \mathbb{C}[\partial]U(\mathfrak{L}_0)u = \mathbb{C}[\partial]u$, and hence $V = \mathbb{C}[\partial]u$ by irreducibility of V . We consider two cases:

CASE 1. Suppose that \mathfrak{L}_0 is an ideal of \mathfrak{L} . Then \mathfrak{L}_0 acts trivially on V . Thus V is an irreducible $\mathbb{C}[\partial]$ -module. It follows that if ∂ is even, then V is 1-dimensional, and if ∂ is odd with $[\partial, \partial] \neq 0$, V is either the trivial or a 2-dimensional $\mathfrak{L}/\mathfrak{L}_0$ -module with $[\partial, \partial]$ acting as a non-zero scalar.

CASE 2. Suppose that \mathfrak{L}_0 is not an ideal of \mathfrak{L} . Then there exist $l_0, l'_0 \in \mathfrak{L}_0$ such that $[l_0, \partial] = \partial + l'_0$. An easy induction argument shows that, $l_0\partial^i u = i\partial^i u + f_i(\partial)u$ for $i \geq 1$, where $f_i(\partial)$ is a polynomial in ∂ of degree at most $i - 1$ with zero constant term. Now if $v = \sum_{i=0}^n a_i \partial^i u$ is a non-zero vector in $\mathbb{C}[\partial] \otimes \mathbb{C}u$ with $a_0 \neq 0$, then $v - (\frac{1}{n})l_0 v$ is a non-zero vector of degree at most $n - 1$. Thus proceeding this way we see that u is contained in the module generated by v . Therefore $\partial\mathbb{C}[\partial] \otimes \mathbb{C}u$ is the unique maximal submodule of $\mathbb{C}[\partial] \otimes \mathbb{C}u$ and hence V is the trivial module. \square

We will now apply the theorem above to classify irreducible conformal modules over the Virasoro algebra, the current algebra and their semidirect product. In addition similar results can be obtained for the corresponding $N=1$ extended superalgebras by slightly modifying the arguments. In order to do so, the following lemma is useful.

Lemma 3.2. [5] *Let \mathfrak{g} be a Lie superalgebra and let \mathfrak{n} be an ideal of \mathfrak{g} . Assume that any finite-dimensional quotient of \mathfrak{n} is solvable. Let \mathfrak{a} be an even subalgebra of \mathfrak{g} such that \mathfrak{n} is a completely reducible $\text{ad}\mathfrak{a}$ -module with no trivial summand. Then \mathfrak{n} annihilates a non-zero vector in any finite-dimensional \mathfrak{g} -module V . In particular \mathfrak{n} acts trivially in any irreducible finite-dimensional \mathfrak{g} -module.*

Proof. First note that if $[a, b] = b$, then b is nilpotent on any finite-dimensional representation V . To see this, note that if $v \in V$ is an eigenvector of a with eigenvalue λ , then the condition $[a, b] = b$ implies that bv is an eigenvector of a with eigenvalue $\lambda + 1$. Thus there exists a non-zero $w \in V$ such that $bw = 0$. Let W be the space annihilated by b . Then W is a -invariant. Since $W \neq 0$, it follows by induction on the dimension of V that b is nilpotent on V/W . Thus b is nilpotent on V .

Taking the image of \mathfrak{g} in $\text{End}_{\mathbb{C}}(V)$, we may assume that $\dim_{\mathbb{C}} \mathfrak{g} < \infty$, hence $\dim_{\mathbb{C}} \mathfrak{a} < \infty$ and \mathfrak{n} is a finite-dimensional solvable Lie superalgebra. Let \mathfrak{s} be an irreducible $\text{ad}\mathfrak{a}$ -submodule of \mathfrak{n} . By the assumption, it is a module with a non-zero highest weight. Hence there exists $a \in \mathfrak{a}$ and $b \in \mathfrak{s}$ such that $[a, b] = b$, therefore b is nilpotent on V . Moreover all elements from the orbit $\text{Ad}A \cdot b$, where A is the connected Lie group with Lie algebra \mathfrak{a} , are nilpotent on V . Since \mathfrak{s} is $\text{Ad}A$ -irreducible, this orbit spans \mathfrak{s} , hence \mathfrak{s} is spanned by elements that are nilpotent on V . Thus any \mathfrak{a} -submodule of \mathfrak{n} is spanned by elements that are nilpotent on V .

To prove the lemma, we may assume that V is a faithful irreducible \mathfrak{g} -module. Suppose the lemma is not true, i.e. \mathfrak{n} is non-zero. Let $\mathfrak{n}^{(i)}$ be the last non-zero member of its derived series. By the above, $\mathfrak{n}^{(i)}$ is spanned by mutually commuting elements that are nilpotent on V , and hence $\mathfrak{n}^{(i)}$ annihilates a non-zero vector in V . But $\mathfrak{n}^{(i)}$ is an ideal of \mathfrak{g} and hence the subspace of V , annihilated by $\mathfrak{n}^{(i)}$, is a \mathfrak{g} -submodule of V . Thus $\mathfrak{n}^{(i)}$ annihilates V and so V is not faithful, which is a contradiction. \square

We are now in a position to classify finite conformal modules over the Virasoro, Neveu-Schwarz, current and the supercurrent algebras and their semidirect sums. Due to Section 2 we only need to classify finite modules over the corresponding (extended) annihilation subalgebras. For each of these Lie superalgebras of formal distributions, the corresponding annihilation subalgebras are of course the corresponding subalgebras defined on the line,

instead of the circle. So we will use terminology like current algebras on the line etc. to denote the corresponding annihilation subalgebras.

The following corollaries are immediate by Theorem 3.1 and Lemma 3.2.

Corollary 3.1. *Let \mathfrak{g} be a direct sum of finite-dimensional simple Lie superalgebras (we allow commutative summands). Let $\mathfrak{L} = \mathbb{C} \frac{d}{dt} \ltimes \mathfrak{g}[t]$, where $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the current algebra on the line. Then every non-trivial irreducible conformal module of \mathfrak{L} is of the form $\text{Ind}_{\mathfrak{g}[t]}^{\mathfrak{L}} U$, where U is a finite dimensional non-trivial irreducible \mathfrak{g} -module or else it is the trivial $\mathfrak{g}[t]$ -module on which $\frac{d}{dt}$ acts as a non-zero scalar.*

Corollary 3.2. *Let \mathfrak{g} be a direct sum of finite-dimensional simple Lie superalgebras. Let $\mathfrak{L} = \mathbb{C} \frac{d}{dt} \ltimes \mathfrak{g}[t, \theta]$, where $\mathfrak{g}[t, \theta] = \mathfrak{g} \otimes \mathbb{C}[t, \theta]$. Then every non-trivial irreducible conformal module of \mathfrak{L} is of the form $\text{Ind}_{\mathfrak{g}[t, \theta]}^{\mathfrak{L}} U$, where U is a finite dimensional non-trivial irreducible \mathfrak{g} -module or else it is the trivial $\mathfrak{g}[t, \theta]$ -module on which $\frac{d}{dt}$ acts as a non-zero scalar.*

Corollary 3.3. *Let $\mathfrak{V}_+ = \sum_{k \geq -1} \mathbb{C} t^{k+1} \frac{d}{dt}$ be the Virasoro algebra on the line. Let $L_i = -t^{i+1} \frac{d}{dt}$ and let $\mathfrak{V}_0 = \sum_{k \geq 0} \mathbb{C} L_k$. Then any non-trivial irreducible conformal module of \mathfrak{V}_+ is of the form $\text{Ind}_{\mathfrak{V}_0}^{\mathfrak{V}_+} \mathbb{C}_\lambda$, where \mathbb{C}_λ is a non-trivial one-dimensional irreducible representation of \mathfrak{V}_0 , on which L_0 acts as $\lambda \in \mathbb{C}^*$ and L_k act as 0 for all $k > 0$.*

Corollary 3.4. *Let $\mathfrak{L} = \mathfrak{V}_+ \ltimes \mathfrak{g}[t]$ such that $[L_k, a \otimes t^n] = -na \otimes t^{n+k}$, $a \in \mathfrak{g}$. Let $\mathfrak{L}_0 = \mathfrak{V}_0 \ltimes \mathfrak{g}[t]$. Then every non-trivial irreducible \mathfrak{L} -module is of the form $\text{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$, where U is a non-trivial irreducible $(\mathfrak{g} \oplus \mathbb{C} L_0)$ -module with $\mathfrak{g}[t]t$ and L_k , $k > 0$, acting trivially.*

Remark 3.1. Translating the modules over the annihilation subalgebra $\mathfrak{V}_+ \ltimes \mathfrak{g}[t]$ of Corollary 3.4 into the language of conformal modules we obtain a 3-parameter family of non-trivial conformal modules over $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. We will denote these modules by $M_{\mathfrak{V} \ltimes \tilde{\mathfrak{g}}}(\Lambda, \alpha, \Delta)$, where Λ stands for the irreducible finite-dimensional \mathfrak{g} -module of highest weight Λ . Clearly when $\Lambda \neq 0$, $M_{\mathfrak{V} \ltimes \tilde{\mathfrak{g}}}(\Lambda, \alpha, \Delta)$ is irreducible. When $\Lambda = 0$, $M_{\mathfrak{V} \ltimes \tilde{\mathfrak{g}}}(0, \alpha, \Delta)$, for $\Delta \neq 0$, is irreducible.

For the Neveu-Schwarz algebra on the line we have the following description:

Corollary 3.5. *Let $\mathfrak{N}_+ = \sum_{n \geq -1} \mathbb{C} L_n + \sum_{r \geq -\frac{1}{2}} \mathbb{C} G_r$ be the Neveu-Schwarz algebra on the line. Let $\mathfrak{N}_0 = \sum_{n \geq 0} \mathbb{C} L_n + \sum_{r > 0} \mathbb{C} G_r$. Then every non-trivial irreducible \mathfrak{N}_+ -module*

is of the form $\text{Ind}_{\mathfrak{N}_0}^{\mathfrak{N}_+} \mathbb{C}_\lambda$, where \mathbb{C}_λ is a one dimensional irreducible representation of \mathfrak{N}_0 , on which L_0 acts as the scalar $\lambda \in \mathbb{C}^*$ and L_k and G_r act trivially for $r, k > 0$.

Proof. We define a filtration on the Lie superalgebra \mathfrak{N}_+ as follows: \mathfrak{N}_i is the subalgebra spanned by $\{L_{\frac{i}{2}}, G_{\frac{i+1}{2}}, L_{\frac{i+2}{2}}, G_{\frac{i+3}{2}}, \dots\}$, if i is even. If i is odd, then \mathfrak{N}_i is spanned by the linearly independent vectors $\{G_{\frac{i}{2}}, L_{\frac{i+1}{2}}, G_{\frac{i+2}{2}}, L_{\frac{i+3}{2}}, \dots\}$. We set $\partial = G_{-\frac{1}{2}}$ so that $[\partial, \partial] = 2L_{-1}$. We then have $\mathfrak{N}_+ = \mathbb{C}\partial + \mathbb{C}[\partial, \partial] + \mathfrak{N}_0$. Hence by Theorem 3.1 every non-trivial irreducible conformal module over \mathfrak{N}_+ is of the form $\text{Ind}_{\mathfrak{N}_0}^{\mathfrak{N}_+} U$, where U is an irreducible \mathfrak{N}_0 -module. Now we use Lemma 3.2 and the result follows. \square

Using similar filtrations as in Corollary 3.5 one proves Corollaries 3.6 and 3.7 below. Although never used, Corollary 3.6 is stated for the sake of completeness.

Corollary 3.6. *Let \mathfrak{g} be a direct sum of finite-dimensional simple Lie superalgebras. Let $\mathfrak{L} = (\mathbb{C}\frac{\partial}{\partial t} + \mathbb{C}(\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta})) \ltimes \mathfrak{g}[t, \theta]$. Then every irreducible non-trivial conformal module of \mathfrak{L} is either of the form $\text{Ind}_{\mathfrak{g}[t, \theta]}^{\mathfrak{L}} U$, where U is a finite-dimensional irreducible representation of \mathfrak{g} or it is an irreducible two dimensional representation of the subalgebra $\mathbb{C}\frac{\partial}{\partial t} + \mathbb{C}(\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta})$, on which $\frac{\partial}{\partial t}$ acts as a non-zero scalar and $\mathfrak{g}[t, \theta]$ acts trivially.*

Corollary 3.7. *Let $\mathfrak{L} = \mathfrak{N}_+ \ltimes \mathfrak{g}[t, \theta]$. Set $\mathfrak{L}_0 = \mathfrak{N}_0 \ltimes \mathfrak{g}[t, \theta]$. Then every non-trivial conformal module over \mathfrak{L} is of the form $\text{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$, where U is a finite-dimensional non-trivial $\mathfrak{g} \oplus \mathbb{C}L_0$ -module, on which L_k, G_r for all $k, r > 0$ and $\mathfrak{g}[t]t + \mathfrak{g}[t]\theta$ act trivially.*

Remark 3.2. Corollary 3.7 gives a 3-parameter family of irreducible conformal modules over $\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}$. As before we will denote the modules over the corresponding conformal superalgebra $R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\text{super}})$ by $M_{\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}}(\Lambda, \alpha, \Delta)$, where Λ is a dominant integral weight of \mathfrak{g} , $\alpha, \Delta \in \mathbb{C}$. The conditions for irreducibility of this module is as in Remark 3.2.

Translating the above corollaries into the language of modules over conformal superalgebras (using Proposition 2.1 and Remark 2.2) we obtain the following

Theorem 3.2. *Let $R(\mathfrak{V})$, $R(\tilde{\mathfrak{g}})$, $R(\mathfrak{N})$ and $R(\tilde{\mathfrak{g}}_{\text{super}})$ stand for the Virasoro, the current, the Neveu-Schwarz and the super current conformal (super)algebras, respectively. Let $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ and $R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\text{super}})$ denote their respective semidirect sums. Then the following is a complete list of their finite non-trivial irreducible modules:*

1. $M_{\mathfrak{V}}(\alpha, \Delta)$ and $M_{\mathfrak{V}}(\alpha, \Delta)$, where $\alpha, \Delta \in \mathbb{C}$ with $\Delta \neq 0$.
2. $M_{\tilde{\mathfrak{g}}}(\Lambda)$ and $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\Lambda)$, where Λ is a non-zero dominant integral weight, or their quotients.
3. $M_{\mathfrak{V} \ltimes \tilde{\mathfrak{g}}}(\Lambda, \alpha, \Delta)$ and $M_{\mathfrak{V} \ltimes \tilde{\mathfrak{g}}_{\text{super}}}(\Lambda, \alpha, \Delta)$, where Λ is a non-zero dominant integral weight and $\alpha, \Delta \in \mathbb{C}$ or else if $\Lambda = 0$, then $\Delta \neq 0$.

It was shown in [2] that every semisimple finite conformal algebra is a direct sum of conformal algebras of the form $R(\mathfrak{V})$, $R(\tilde{\mathfrak{g}})$ and $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. The results of this section give a description of all finite irreducible modules over all finite semisimple conformal algebras. Namely we have the following

Proposition 3.1. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ be a finite semisimple conformal algebra. Suppose that R_i is either $R(\mathfrak{V})$ or $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ for some i . Let V be a finite irreducible conformal module of R whose restriction to R_i is non-trivial. Then the restriction of V to all R_j is trivial for $i \neq j$.*

Proof. Since R_i is either the conformal algebra $R(\tilde{\mathfrak{g}})$ or $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$, there exists $L_{(0)}^i \in R_i$ such that $[\partial - L_{(0)}^i, R_i] = 0$. Choose any $k \neq i$ and consider irreducible modules over the conformal algebra $R_i \oplus R_k$. By Proposition 2.1 we are to consider modules over $\mathbb{C}\partial \ltimes ((R_i)_+ \oplus (R_k)_+) \cong (R_i)_+ \oplus (\mathbb{C}(\partial - L_{(0)}^i) \ltimes (R_k)_+)$. Now a non-trivial irreducible $(R_i)_+$ -module is free over $\mathbb{C}[L_{(0)}^i]$ and a non-trivial irreducible $\mathbb{C}(\partial - L_{(0)}^i) \ltimes (R_k)_+$ -module is free over $\mathbb{C}[\partial - L_{(0)}^i]$ by above discussion. Thus their tensor product is free over $\mathbb{C}[L_{(0)}^i] \otimes \mathbb{C}[\partial - L_{(0)}^i]$ and hence cannot be finite over $\mathbb{C}[\partial]$. \square

Proposition 3.1 and Theorem 3.2 imply immediately

Theorem 3.3. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ be a finite semisimple conformal algebra, where each R_i is either simple or of the form $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. Then R has a faithful finite module if and only if either all R_i are current conformal algebras or $n = 1$.*

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Erratum: Conformal Modules

by

Shun-Jen Cheng[†] and Victor G. Kac^{††}

[†]Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan
chengsj@mail.ncku.edu.tw

^{††}Department of Mathematics, MIT, Cambridge, MA 02139, USA
kac@math.mit.edu

In this erratum we make corrections (a),(b) and (c) to our paper [3].

(a) Here is a correct statement and a proof of Lemma 3.2 of [3].

Lemma 1. *Let \mathfrak{g} be a finite-dimensional Lie superalgebra and let \mathfrak{n} be a solvable ideal of \mathfrak{g} . Let \mathfrak{a} be an even subalgebra of \mathfrak{g} such that \mathfrak{n} is a completely reducible $\text{ad}\mathfrak{a}$ -module with no trivial summand. Then \mathfrak{n} acts trivially in any irreducible finite-dimensional \mathfrak{g} -module V .*

Proof. First note that if $[a, b] = b$, then b is nilpotent on any finite-dimensional representation V . To see this, note that if $v \in V$ is an eigenvector of a with eigenvalue λ , then the condition $[a, b] = b$ implies that bv is an eigenvector of a with eigenvalue $\lambda + 1$. Thus there exists a non-zero $w \in V$ such that $bw = 0$. Let W be the space annihilated by b . Then W is a -invariant. Since $W \neq 0$, it follows by induction on the dimension of V that b is nilpotent on V/W . Thus b is nilpotent on V .

Let \mathfrak{s} be an irreducible $\text{ad}\mathfrak{a}$ -submodule of \mathfrak{n} . By the assumption, it is a module with a non-zero highest weight. Hence there exists $a \in \mathfrak{a}$ and $b \in \mathfrak{s}$ such that $[a, b] = b$, therefore b is nilpotent on V . Moreover all elements from the orbit $\text{Ad}A \cdot b$, where A is the connected Lie group with Lie algebra \mathfrak{a} , are nilpotent on V . Since \mathfrak{s} is $\text{Ad}A$ -irreducible, this orbit spans \mathfrak{s} , hence \mathfrak{s} is spanned by elements that are nilpotent on V . Thus any \mathfrak{a} -submodule of \mathfrak{n} is spanned by elements that are nilpotent on V .

To prove the lemma, we may assume that V is a faithful \mathfrak{g} -module. Suppose the lemma is not true, i.e. \mathfrak{n} is non-zero. Let $\mathfrak{n}^{(i)}$ be the last non-zero member of its derived series. By the above, $\mathfrak{n}^{(i)}$ is spanned by mutually commuting elements that are nilpotent

on V , and hence $\mathfrak{n}^{(i)}$ annihilates a non-zero vector in V . But $\mathfrak{n}^{(i)}$ is an ideal of \mathfrak{g} and hence the subspace of V , annihilated by $\mathfrak{n}^{(i)}$, is a \mathfrak{g} -submodule of V . Thus $\mathfrak{n}^{(i)}$ annihilates V and so V is not faithful, which is a contradiction. \square

(b) Lemma 1, however, is not applicable to the current algebra of three series of simple Lie superalgebras that we have considered in [3]. Namely the Lie superalgebras $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$ (see [4]) for which \mathfrak{n} will contain trivial summands. For these three series the statement of Corollary 3.1 and Theorem 3.2 turns out to be incorrect. Below we will classify irreducible conformal modules over their current algebras. Due to Lemma 3.1 of [3] it suffices to consider finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]/t^{n+1}$, where $n \geq 0$ and \mathfrak{g} is a member of one of the three series of simple Lie superalgebras above. As the case of $n = 0$ is trivial, we may assume from now on that $n \geq 1$. In fact we will study more general Lie superalgebras \mathfrak{g} satisfying properties we now describe.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra and suppose that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ is \mathbb{Z} -graded such that $\mathfrak{g}_i \subseteq \mathfrak{g}_{\bar{i}}$. Assume that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathbb{C}c$ is a reductive Lie algebra such that \mathfrak{a} is a semisimple subalgebra and c is a central element. Furthermore suppose that \mathfrak{g}_i as an \mathfrak{a} -module has no trivial summand for $i \neq 0$, and there exists an \mathfrak{a} -submodule $\mathfrak{g}_{-1}^* \subseteq \mathfrak{g}_1$ contragredient to \mathfrak{g}_{-1} and $\mathfrak{g}_1 = \mathfrak{g}_{-1}^* \oplus \mathfrak{g}'_1$ as \mathfrak{a} -modules with $[\mathfrak{g}'_1, \mathfrak{g}_{-1}] \subseteq \mathfrak{a}$. Finally suppose that given any non-zero $a \in \mathfrak{g}_{-1}$ $[a, \mathfrak{g}_{-1}^*] \cap \mathbb{C}c \neq 0$ and given any non-zero $b \in \mathfrak{g}_{-1}^*$ $[\mathfrak{g}_{-1}, b] \cap \mathbb{C}c \neq 0$. Note that it follows from the descriptions of the three series of simple Lie superalgebras $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$ in [4] that they satisfy the assumptions of \mathfrak{g} above.

Set $\mathfrak{L} = \mathfrak{g} \otimes \mathbb{C}[t]/t^{n+1}$. We want to determine finite-dimensional irreducible \mathfrak{L} -modules, on which $\mathfrak{g} \otimes t^n$ acts non-trivially. We let

$$G_0 = \mathfrak{g}_0 + \mathfrak{g}'_1 + \sum_{i \geq 2} \mathfrak{g}_i.$$

Consider the subalgebra $L \subseteq \mathfrak{L}$, which is defined as follows: For $n = 2k$, $k \in \mathbb{N}$, an even integer, we let

$$L := G_0 + G_0 \otimes t + \cdots + G_0 \otimes t^{k-1} + (G_0 + \mathfrak{g}_1) \otimes t^k + \mathfrak{g} \otimes t^{k+1} + \cdots + \mathfrak{g} \otimes t^{2k}.$$

If $n = 2k + 1$ is odd, then we let

$$L := G_0 + G_0 \otimes t + \cdots + G_0 \otimes t^k + \mathfrak{g} \otimes t^{k+1} + \cdots + \mathfrak{g} \otimes t^{2k+1}.$$

We first determine finite-dimensional irreducible L -modules. It will turn out that every irreducible \mathfrak{L} -module on which $\mathfrak{g} \otimes t^n$ acts non-trivially are obtained from inducing from a suitable irreducible L -module. The main tool we use to prove this assertion is the Lie algebraic analogue of Mackey's irreducibility criterion [1]. Before recalling it, we need some terminology. Let \mathfrak{k} be a finite-dimensional Lie superalgebra and $I \subseteq \mathfrak{k}$ be an ideal of \mathfrak{k} . Let (π, V_I) be an irreducible I -module. Define the stabilizer associated to the pair (π, I) to be $K_\pi = \{k \in \mathfrak{k} | \exists A_k \in \text{End}(V_I) \text{ with } \pi([k, i]) = [A_k, \pi(i)], \forall i \in I\}$. K_π is a subalgebra containing I . One can prove using analogous arguments as in [1]

Theorem 1. [2] *Let V_K be an irreducible representation of K_π such that as an I -module, V_K is a direct sum of copies of π . If the \mathbb{Z}_2 -graded space \mathfrak{k}/K_π is spanned by elements of the same parity, then $\text{Ind}_{K_\pi}^{\mathfrak{k}} V_K$ is an irreducible \mathfrak{k} -module.*

We now consider the subalgebra

$$B = \mathfrak{b} + \mathfrak{g}'_1 + \sum_{i \geq 2} \mathfrak{g}_i + (L \cap \mathfrak{g} \otimes \mathbb{C}[t]t),$$

where \mathfrak{b} is a Borel subalgebra of \mathfrak{g}_0 . Then B is a solvable subalgebra of L and $[B_{\bar{1}}, B_{\bar{1}}] \cap \mathbb{C}c = 0$. However, since \mathfrak{g} as an \mathfrak{a} -module contains only one trivial summand, namely $\mathbb{C}c$, it is clear that $[\mathfrak{b}, \mathfrak{g}_i] = \mathfrak{g}_i$, for all $i \neq 0$, and $[\mathfrak{b}, \mathfrak{g}_0] = \mathfrak{a}$. In fact $[\mathfrak{b}, \mathfrak{g}_{-1}^*] = \mathfrak{g}_{-1}^*$ and $[\mathfrak{b}, \mathfrak{g}'_1] = \mathfrak{g}'_1$. This discussion implies that $[B_{\bar{1}}, B_{\bar{1}}] \subseteq [B_{\bar{0}}, B_{\bar{0}}]$. Therefore by [4] every finite-dimensional irreducible representation of B is one-dimensional, say $\mathbb{C}v_\lambda$, where $\lambda : \mathfrak{h} + \mathbb{C}c \otimes \mathbb{C}[t] \rightarrow \mathbb{C}$ is a linear form extended trivially to B and \mathfrak{h} is a Cartan subalgebra in \mathfrak{b} . However $\text{Ind}_B^L \mathbb{C}v_\lambda$ is spanned by elements of the form $e_{-\alpha_1}^{k_{\alpha_1}} \cdots e_{-\alpha_m}^{k_{\alpha_m}} v_\lambda$, where $e_{-\alpha_j}$ are negative root vectors of \mathfrak{a} . It follows that $c \otimes t^j$ acts as a scalar on $\text{Ind}_B^L \mathbb{C}v_\lambda$. Set $\mathfrak{g}^c = \mathfrak{g}_{-1} + \mathfrak{a} + \sum_{i \geq 1} \mathfrak{g}_i$. Clearly $[\mathfrak{a}, \mathfrak{g}^c] = \mathfrak{g}^c$. From this it follows by induction on $k_{\alpha_1} + \cdots + k_{\alpha_m}$ that $\mathfrak{g}'_1 + \sum_{i \geq 2} \mathfrak{g}_i + (\mathfrak{g}^c \otimes \mathbb{C}[t]t) \cap L$ acts trivially on this induced module. Thus we obtain

Proposition 1. *Every irreducible L -module is an irreducible $\mathfrak{g}_0 \oplus (c \otimes \mathbb{C}[t]t)$ -module, on which $\mathfrak{g}'_1 + \sum_{i \geq 2} \mathfrak{g}_i + (\mathfrak{g}^c \otimes \mathbb{C}[t]t) \cap L$ acts trivially.*

Next we introduce the auxiliary subalgebra

$$\bar{\mathfrak{L}} := G_0 + \mathfrak{g} \otimes \mathbb{C}[t]t.$$

From Proposition 1 we obtain the classification of finite-dimensional irreducible $\bar{\mathfrak{L}}$ -modules.

Proposition 2. *Let V_L be an irreducible L -module such that $c \otimes t^n$ acts as a non-zero scalar. Then $\text{Ind}_L^{\bar{\mathfrak{L}}} V_L$ is irreducible. Furthermore suppose that W is an irreducible $\bar{\mathfrak{L}}$ -module on which $\mathfrak{g} \otimes t^n$ acts non-trivially, then $W \cong \text{Ind}_L^{\bar{\mathfrak{L}}} V_L$, where V_L is as above.*

Proof. Let $I = (G_0 + \mathfrak{g}_1) \otimes t^k + \mathfrak{g} \otimes \mathbb{C}[t]t^{k+1}$, if $n = 2k$ and $I = \mathfrak{g} \otimes \mathbb{C}[t]t^{k+1}$, if $n = 2k + 1$. Note that I is an ideal of $\bar{\mathfrak{L}}$. By Proposition 1 V_L is a direct sum of 1-dimensional mutually isomorphic I -modules $\mathbb{C}v_\lambda$, where $(c \otimes t^j)v_\lambda = \lambda(c \otimes t^j)v_\lambda$. Consider the stabilizer of the I -module $\mathbb{C}v_\lambda$ in $\bar{\mathfrak{L}}$. Since for every vector $g \in \mathfrak{g}_{-1} + \mathfrak{g}_{-1}^*$ there exists a vector $g' \in \mathfrak{g}_{-1} + \mathfrak{g}_{-1}^*$ such that $[g, g'] = \mu c + a$, where $\mu \neq 0$ and $a \in \mathfrak{a} + \mathfrak{g}_2$, it follows that the stabilizer is precisely L , if $\lambda(c \otimes t^n) \neq 0$. Since also $\bar{\mathfrak{L}}/L$ is a completely odd vector space, we can employ Theorem 1 and irreducibility follows. On the other hand suppose that W is an irreducible $\bar{\mathfrak{L}}$ -module on which $\mathfrak{g} \otimes t^n$ acts non-trivially. Let V_L be an irreducible L -submodule inside W . If $c \otimes t^n$ acts trivially on V_L , then $\mathfrak{g} \otimes t^n$ acts trivially on V_L by Proposition 1. But $\mathfrak{g} \otimes t^n$ is an ideal of $\bar{\mathfrak{L}}$ and hence acts trivially on W , which is a contradiction. \square

Proposition 1 says that inducing from L to $\bar{\mathfrak{L}}$ irreducibility is preserved. So our final goal is to show that induction from $\bar{\mathfrak{L}}$ to \mathfrak{L} also preserves irreducibility. Note that from Proposition 2 we get

Corollary 1. *Every irreducible $\bar{\mathfrak{L}}$ -module, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, considered as a $\mathfrak{g} \otimes t^n$ -module, is a direct sum of the same 1-dimensional $\mathfrak{g} \otimes t^n$ -module $\mathbb{C}v_\lambda$. Furthermore $c \otimes t^n$ acts as a non-zero scalar $\lambda(c \otimes t^n)$, while $\mathfrak{g}^c \otimes t^n$ acts trivially.*

Now we are ready to classify irreducible \mathfrak{L} -modules.

Proposition 3. *Let V be an irreducible $\bar{\mathfrak{L}}$ -module on which $c \otimes t^n$ acts as a non-zero scalar. Then $\text{Ind}_{\bar{\mathfrak{L}}}^{\mathfrak{L}} V$ is an irreducible \mathfrak{L} -module. Furthermore every irreducible \mathfrak{L} -module, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, is of this form.*

Proof. We will set up to employ Theorem 1 again. By Corollary 1 V is a direct sum of copies of the $\mathfrak{g} \otimes t^n$ -module $\mathbb{C}v_\lambda$ and $\lambda(c \otimes t^n) \neq 0$. We compute the stabilizer of the $\mathfrak{g} \otimes t^n$ -module $\mathbb{C}v_\lambda$ now in \mathfrak{L} . Again using the same argument as in the proof of Proposition 2, one checks that the stabilizer is precisely $\bar{\mathfrak{L}}$, if $\lambda(c \otimes t^n) \neq 0$. Thus by Blattner's theorem again, the induced module is irreducible. The same argument as in Proposition 2 gives that these are all such irreducibles. \square

Combining Proposition 2 and 3 gives

Theorem 2. *All finite-dimensional irreducible representations of \mathfrak{L} , on which $\mathfrak{g} \otimes t^n$ acts non-trivially, are of the form $\text{Ind}_L^{\mathfrak{L}} V_L$, where V_L is an irreducible representation of L , on which $c \otimes t^n$ acts as a non-zero scalar. Furthermore all such representations are irreducible.*

In [2] it was shown that if S is a finite-dimensional Lie superalgebra such that $[S, S] = S$ and $\Lambda(n)$ is the Grassmann superalgebra in n odd variables, then every finite-dimensional irreducible $S \otimes \Lambda(n)$ -module is an irreducible S -module, on which the $S \otimes N$ acts trivially, where N is the maximal nilpotent ideal of $\Lambda(n)$. Thus for supercurrents $\mathfrak{g}[t, \theta] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(1)$, where θ is an odd variables and \mathfrak{g} is one of the three series of simple Lie superalgebras, we may apply this result and Theorem 2 to obtain their irreducible conformal modules.

(c) In the proof of Lemma 3.1 in [3] we use v to denote the vector $\sum_{i=1}^n p_i(\partial)v_i$.

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